What is the relationship between the initial temperature of water and its rate of cooling?

Introduction

During the warm summers of Dubai, I love making homemade ice cream. Once, my impatience got the better of me, and I placed the warm mixture in the freezer without letting it cool. What I found was that it froze faster than usual. Being curious, I later attempted an informal experiment where I placed two cream mixtures at different temperatures in the freezer to see which one would freeze faster. The warm mixture cooled faster again. I did a little research and found that I was following in the footsteps of Erasto Mpemba, who observed this in the same situation. I had to confirm this mystery for myself, so I decided to perform a formal experiment.

In my experiment, I aim to find a relationship between the initial temperature of water and its rate of cooling. I hope to find an equation different from that of Newton's Law of Cooling which would account for the Mpemba effect.

Background Information

Newton's Law of Cooling is a relatively well-known concept in the world of physics. It is represented by the formula

$$\frac{dT}{dt} = -k(T - T_s)$$

Where t is time, T is the temperature of the substance at time t, $\frac{dT}{dt}$ is the rate of cooling, k is the constant heat transfer coefficient, and $(T - T_s)$ is the difference between the temperature of the substance and its surroundings at time t.

According to this law, the rate at which an object cools is proportional to the difference in temperature between the object and its surroundings. However, an irregularity in the law was discovered by Erasto Mpemba, who observed that warm water freezes faster than cool water. The "Mpemba Effect" is one of the supposed big seven paradoxes of thermodynamics. Paradoxes in scientific theories intrigue me, and I wanted to investigate one of them to hopefully shine some light on how we could enhance our understanding of the world and explain such paradoxes.

An interesting example of how this effect is used is in making ice cream. For a long time, many ice cream makers have made their ice cream using warmed milk since it can be made to freeze faster than cold milk[i].

There are many mechanisms theorized as responsible for the Mpemba effect, but none have been confirmed yet. Some mechanisms relevant to my experiment are:

- Frost Melting - When placing the beakers in a frosty freezer, the frost acts as a thermal insulator. However, the warm beaker might melt the frost around it, thus creating a conductive layer of liquid water around the beaker, dramatically increasing its rate of cooling[ii].
- Dissolved Gasses - The rate of cooling of water changes depending on the substances dissolved in it, including gases. Hot water has fewer gases dissolved in it compared to cold water, and this might be one cause of the Mpemba effect.
- Evaporation - When hot water is placed in a freezer, some of it would evaporate before the setup starts to cool. Since the water evaporates, there is less of it in the beaker to cool, allowing it to possibly cool faster than initially colder water. In my experiment, I explored the significance of evaporation on the rate of cooling.
- Convection - As the water cools it will eventually develop convection currents and a non-uniform temperature distribution.[iii] Certain scientists believe that these convection currents, which develop in water that has cooled, could be one of the causes of the Mpemba effect.

Variables

To investigate the relationship between the initial temperature of water and its rate of cooling, I needed to identify certain variables that I would change, the ones that I would expect to be affected by this change, and the ones I would have to keep constant.

Independent Variable

The **initial temperature of water** (measured in °C) in the beaker was varied by heating it to the temperature required for that trial. I did not heat the water separately from the beaker because if I did that, the temperature difference between the beaker and the water would cause the water to start cooling once placed in the beaker, thus causing a larger uncertainty in the experiment.

Dependent Variable

The **time taken for water to cool** (measured in seconds) differed depending on the initial temperature of water. According to Newton's Law of Cooling, water initially at a lower temperature should reach 4°C faster, but this is contested by Mpemba's findings.

Control Variables

The **surrounding temperature** (measured in °C) was kept constant by placing the setup in the freezer. However, because of the way freezers work, their temperature tends to fluctuate slightly. This lead to a significant uncertainty in the experiment that would have to be considered while processing data. By taking three trials for each initial temperature, this uncertainty can be minimized.

The **starting mass** (measured in grams) **and volume of water used** (measured in milliliters) were kept constant for all trials of the experiment. I used a graduated beaker to ensure that 200ml of water is used every time and weighed the setup to ensure that the mass of water is constant between trials.

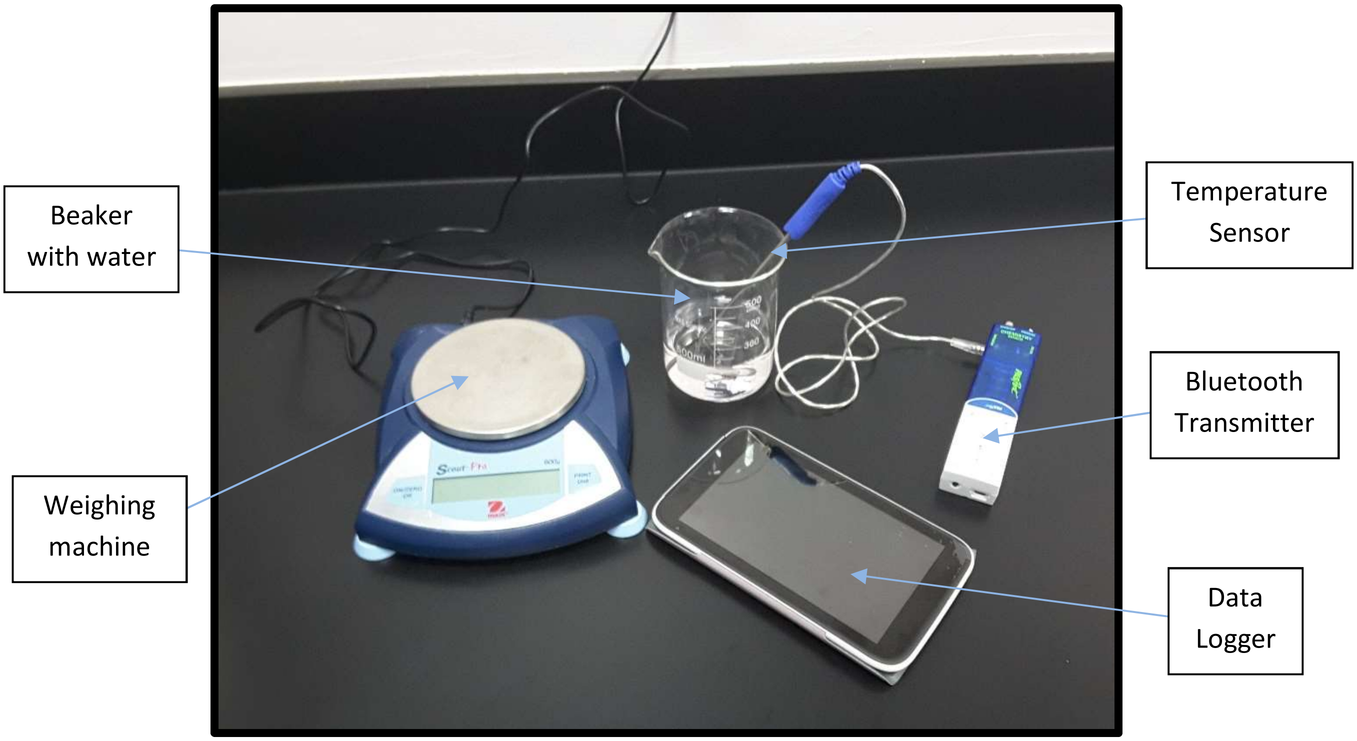
Materials Required

To conduct this experiment, I required a few specific items. I needed a **beaker filled with tap water** and a **weighing machine** to measure the weight of the setup before and after the experiment. I used tap water for the experiment because a common theory suggests that the Mpemba effect is affected by dissolved gasses and impurities in the water.

I couldn’t place the data logger directly in the freezer as it could damage the internal electronics, so I had to use this wireless setup. I required a **Bluetooth temperature sensor** and a corresponding **data logger** for it to wirelessly connect to.

To heat the setup to the required temperature, I needed a **Bunsen burner**. For this step, I required the aid of a lab assistant for this step as it brought up safety concerns.

Experimental Setup



Method

I measured the weight of the empty beaker and then put 200ml of water in it. I always used the same source of water so that the gases dissolved in it stays relatively constant. Since this experiment is being conducted in a high school laboratory, it would be difficult to control the amount of dissolved gases by any other method. I heated the water to the temperature required for that trial. Both the water and the beaker needed to be at the same temperature at the start of the experiment.

I measured the weight of the beaker again to find the mass of water in it. I did this after heating the water because some water might have evaporated during the heating process, causing the mass of water in the beaker to change. I placed the wireless temperature sensor in the beaker and put the entire setup in the freezer, ensuring to place it in the same spot in the freezer for all trials so that the surrounding temperature is the same. I insulated the base of the beaker from the freezer to prevent frost melting.

I set the data logger to record the temperature of the setup every second. Once the data logger read 3°C, I removed the setup from the freezer. I measured the weight of the setup after the experiment to check if any water was lost by evaporation in the freezer.

I repeated the previous steps for the selected temperatures (24°C,34°C,44°C, 54°C, and 64°C), performed each trial 3 times, and took the average of the trials to reduce random uncertainties. Finally, I copied the data from the data logger and processed the data.

Raw Data

Below is the data gathered from the experiment, averaged over 3 trials for each starting temperature. I am using data at discrete intervals of temperature instead of discrete intervals of time because I find it much easier to handle and interpret this data. Unprocessed data can be found in the appendix as there is too much to include in the main report. The uncertainty in the time taken to reach a certain temperature was found by the formula $\Delta t = \frac{t_{max} - t_{min}}{2}$ for each trial.

For example, to calculate the uncertainty in time for the starting temperature of 24°C, I found a data point with the greatest difference between trials and used the formula.

$$\Delta t = \frac{2191 - 1950}{2} = 120\ s$$

I repeated this process for all starting temperatures. The data tables below outline how much time it took for the sample of water to reach the specified temperature.

Cooling from 24 °C to 4 °C					
Temperature (°C) ± 0.1°C	Time (s)	ΔTime (s)	Temperature (°C) ± 0.1°C	Time (s)	ΔTime (s)
24.0	0	± 0	13.0	697	± 104
23.0	24	± 4	12.0	776	± 98
22.0	62	± 7	11.0	861	± 101
21.0	113	± 16	10.0	944	± 105
20.0	173	± 30	9.0	1036	± 101
19.0	236	± 53	8.0	1227	± 108
18.0	306	± 76	7.0	1486	± 121
17.0	379	± 103	6.0	1734	± 119
16.0	449	± 107	5.0	2048	± 121
15.0	525	± 109	4.0	2285	± 119
14.0	610	± 106			

Cooling from 34 °C to 4 °C					
Temperature (°C) ± 0.1°C	Time (s)	ΔTime (s)	Temperature (°C) ± 0.1°C	Time (s)	ΔTime (s)
34.0	0	± 0	18.0	926	± 233
33.0	18	± 1	17.0	1062	± 259
32.0	44	± 6	16.0	1210	± 200
31.0	74	± 5	15.0	1378	± 223
30.0	109	± 9	14.0	1550	± 200
29.0	149	± 11	13.0	1759	± 219
28.0	191	± 11	12.0	1894	± 248
27.0	236	± 12	11.0	1983	± 247
26.0	284	± 10	10.0	2069	± 250
25.0	332	± 15	9.0	2165	± 249
24.0	382	± 11	8.0	2253	± 248
23.0	432	± 2	7.0	2353	± 247
22.0	482	± 4	6.0	2446	± 246
21.0	555	± 24	5.0	2549	± 256
20.0	670	± 71	4.0	2649	± 270
19.0	789	± 148			

Cooling from 44 °C to 4 °C					
Temperature (°C) ± 0.1°C	Time (s)	ΔTime (s)	Temperature (°C) ± 0.1°C	Time (s)	ΔTime (s)
44.0	0	± 0	23.0	1130	± 88
43.0	24	± 4	22.0	1210	± 88
42.0	61	± 3	21.0	1289	± 94
41.0	101	± 8	20.0	1340	± 90
40.0	138	± 11	19.0	1393	± 89
39.0	169	± 8	18.0	1451	± 89
38.0	207	± 16	17.0	1508	± 88
37.0	252	± 19	16.0	1568	± 95
36.0	300	± 34	15.0	1628	± 89
35.0	333	± 38	14.0	1686	± 95
34.0	367	± 38	13.0	1752	± 91
33.0	412	± 28	12.0	1819	± 90
32.0	457	± 36	11.0	1888	± 94
31.0	515	± 56	10.0	1955	± 93
30.0	564	± 68	9.0	2029	± 96
29.0	639	± 89	8.0	2100	± 97
28.0	692	± 87	7.0	2173	± 91
27.0	758	± 87	6.0	2245	± 94
26.0	851	± 87	5.0	2313	± 88
25.0	944	± 88	4.0	2382	± 92
24.0	1050	± 87			

Cooling from 54 °C to 4 °C

Temperature (°C) ± 0.1°C	Time (s)	ΔTime (s)	Temperature (°C) ± 0.1°C	Time (s)	ΔTime (s)
54.0	0	± 0	28.0	770	± 61
53.0	16	± 1	27.0	808	± 57
52.0	42	± 2	26.0	851	± 58
51.0	69	± 5	25.0	892	± 62
50.0	94	± 9	24.0	938	± 62
49.0	119	± 5	23.0	983	± 55
48.0	144	± 12	22.0	1031	± 57
47.0	167	± 19	21.0	1079	± 56
46.0	193	± 34	20.0	1131	± 58
45.0	219	± 37	19.0	1181	± 61
44.0	246	± 36	18.0	1234	± 60
43.0	271	± 34	17.0	1289	± 56
42.0	299	± 34	16.0	1345	± 58
41.0	328	± 37	15.0	1406	± 59
40.0	357	± 35	14.0	1464	± 60
39.0	386	± 33	13.0	1527	± 54
38.0	417	± 32	12.0	1590	± 54
37.0	446	± 27	11.0	1662	± 50
36.0	477	± 25	10.0	1742	± 48
35.0	512	± 31	9.0	1827	± 41
34.0	549	± 30	8.0	1917	± 38
33.0	579	± 36	7.0	2012	± 42
32.0	617	± 43	6.0	2112	± 39
31.0	653	± 47	5.0	2222	± 41
30.0	690	± 47	4.0	2335	± 38
29.0	731	± 56			

Cooling from 64 °C to 4 °C

Temperature (°C) ± 0.1°C	Time (s)	ΔTime (s)	Temperature (°C) ± 0.1°C	Time (s)	ΔTime (s)
64.0	0	± 0	33.0	938	± 59
63.0	11	± 2	32.0	975	± 60
62.0	31	± 6	31.0	1013	± 63
61.0	63	± 7	30.0	1050	± 63
60.0	102	± 10	29.0	1090	± 62
59.0	144	± 23	28.0	1132	± 64
58.0	193	± 23	27.0	1172	± 61
57.0	232	± 30	26.0	1216	± 64
56.0	272	± 36	25.0	1260	± 63
55.0	313	± 46	24.0	1305	± 61
54.0	351	± 52	23.0	1351	± 61
53.0	391	± 61	22.0	1399	± 61
52.0	413	± 64	21.0	1448	± 60
51.0	429	± 60	20.0	1501	± 60
50.0	449	± 58	19.0	1552	± 62
49.0	472	± 61	18.0	1608	± 62
48.0	495	± 61	17.0	1661	± 64
47.0	519	± 64	16.0	1888	± 64
46.0	544	± 61	15.0	2052	± 61
45.0	568	± 64	14.0	2203	± 62
44.0	597	± 64	13.0	2384	± 62
43.0	624	± 59	12.0	2570	± 64
42.0	651	± 62	11.0	2794	± 61
41.0	681	± 63	10.0	3030	± 63
40.0	710	± 64	9.0	3194	± 61
39.0	740	± 63	8.0	3259	± 63
38.0	771	± 58	7.0	3341	± 65
37.0	802	± 60	6.0	3430	± 60
36.0	835	± 62	5.0	3527	± 63
35.0	869	± 61	4.0	3616	± 63
34.0	904	± 65			

Data Processing

Once I gathered all this data, I had to choose what data to use and apply it to the equation.

The properties of water differ between 0°C and 4°C, as compared to other temperatures. For example, water starts to expand once it's temperature is below 4°C. I am not sure whether these differing properties would have an impact on water's rate of cooling, so I am considering only data points above 4°C while processing the data.

I need to manipulate Newton's Law of Cooling before being able to use it[iv]. As a reminder, this is the equation

$$\frac{dT}{dt} = -k(T - T_s)$$

It initially seemed difficult to find a solution for this differential equation, but then I realized that it resembled the formula for radioactive decay. Applying my knowledge from the Quantum and Nuclear Physics unit, I managed to solve the equation[v].

First, I rearranged the equation

$$\frac{dT}{(T - T_s)} = -kdt$$

Integrating both sides of the equation gives

$$\ln(T - T_s) = -kt + \ln C$$

I then used the rules of exponents and logarithms to find a solution to the equation

$$\ln(T - T_s) = \ln(e^{-kt}) + \ln C$$

$$\ln(T - T_s) = \ln(Ce^{-k} \ )$$

$$T - T_s = Ce^{-kt}$$

To find C, I considered the equation at time $t = 0$. Here, $T = T_0$. Substituting into the equation gives $C = T_0 - T_s$

$$T - T_s = (T_0 - T_s)e^{-kt}$$

And finally, rearranging the equation gives us

$$T = T_s + (T_0 - T_s)e^{-kt}$$

Now, I shall use this formula to calculate the value for k in the initial equation and then predict the time it would take for water at different initial temperatures to cool till 4°C. To calculate the unknown value of k, I shall use the trial where the initial temperature of the water was 24°C because the water wasn't heated before the experiment. Hence, there should be no irregularity influencing the data collected (such as a low amount of dissolved gases, fast convection currents in the water, etc.) and I should arrive at an accurate value of k. This value of k would be universal for all other trials as all variables (such as exposed surface area) other than starting temperature are held constant.

The values are

$$T = 4, T_s = -18, T_0 = 24, t = 2285$$

Substituting into the equation gives

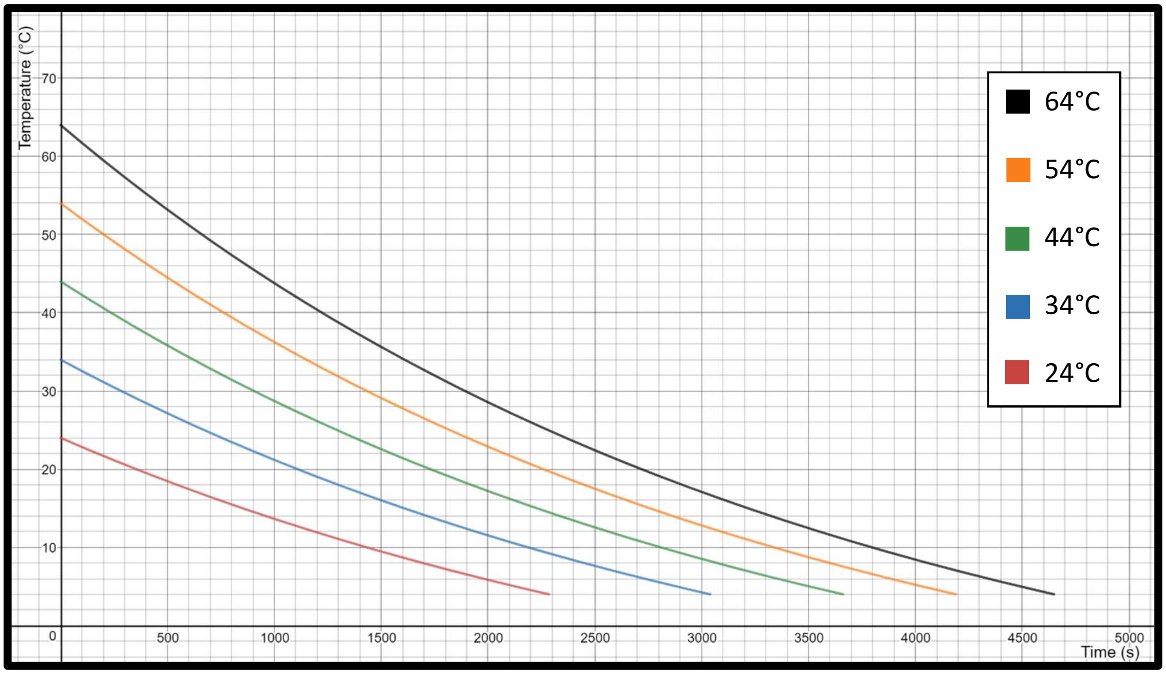
$$4 = -18 + (24 + 18)e^{-2285k} = -18 + (42)e^{-2285k}$$

Solving for k, I get $k = 0.000283$

Hence, according to Newton's Law, the rate of cooling is modeled by the equations

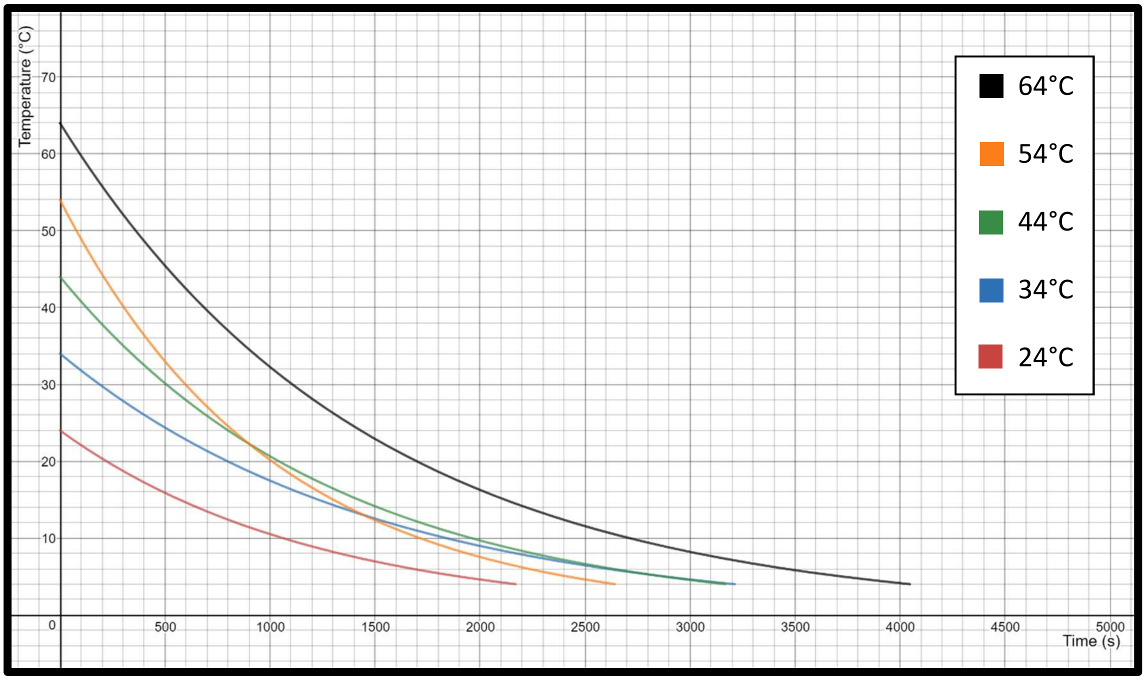
$$T = (T_0 + 18)e^{-0.000283t} - 18$$

Plotting the equations for each initial temperature, I got the graph below.



Graph 1.1 - Expected time taken based on Newton's Law of Cooling

This graph predicts that the cool water reaches 4°C first. However, this isn't supported by my experimental data or Mpemba's findings. Below are the graphs of the data obtained from the experiment conducted. (Note: the error bars have not been included in these graphs for the sake of neatness. Graphs with error bars can be found in the appendix)



Graph 1.2 - Actual time taken

Comparing the values predicted by Newton's Law of Cooling to the experimental values, there is a large difference. The water initially at 54°C cools faster than water starting at 34°C or 44°C and the water initially at 44°C cools faster than water at 34°C. However, the water starting at 24°C still cools fastest, and water starting at 64°C cools the slowest. I can conclude that there is a specific range of temperatures for which the Mpemba effect can be observed.

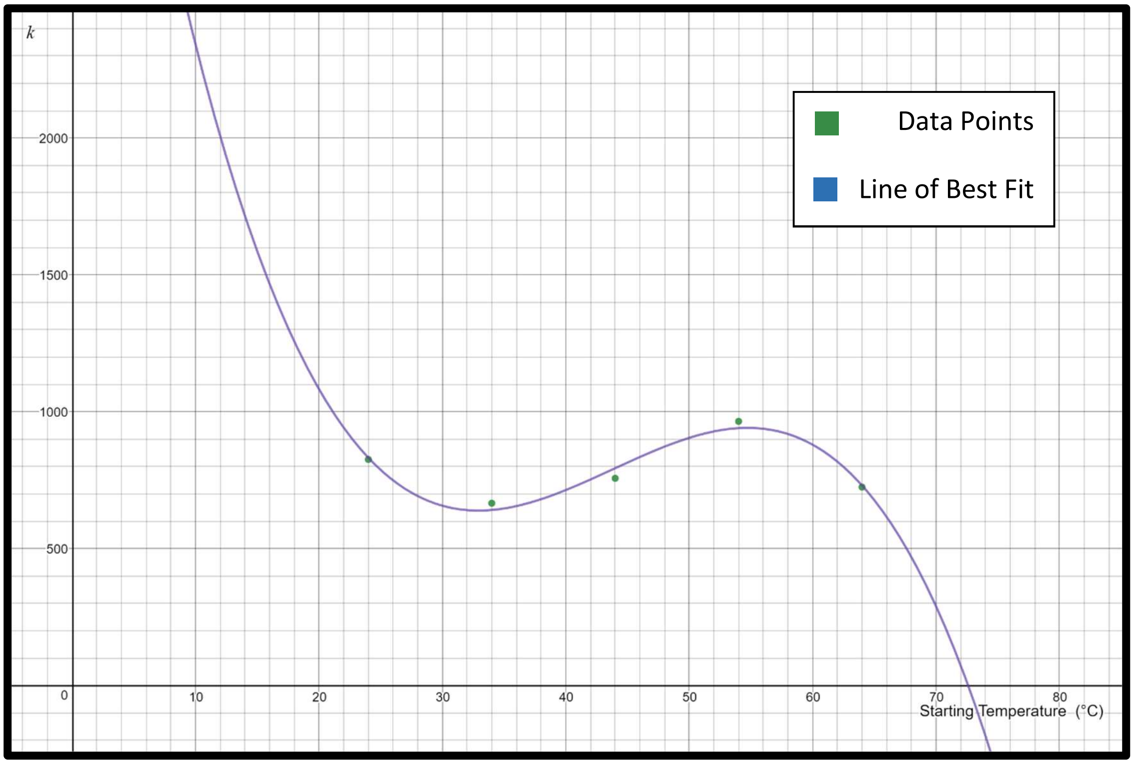
Using computer software, I found the equation of best fit for the data. I now had the following table.

Starting Temperature (°C)	Equation of Best Fit	k	Expected Time Taken (s) (By Newton's Law of Cooling)	Actual Time Taken (s)
24	$24e^{-0.000826t}$	0.000826	2285	2285 ± 119
34	$34e^{-0.000666t}$	0.000666	3039	2649 ± 270
44	$44e^{-0.000757t}$	0.000757	3661	2382 ± 92
54	$54e^{-0.000965t}$	0.000965	4189	2335 ± 38
64	$64e^{-0.000685t}$	0.000685	4646	3616 ± 63

An assumption for Newton's Law of Cooling is that the heat transfer coefficient k is constant. However, the table above indicates that the value changes - the power of e is different for each starting temperature. I hypothesize that k depends on the temperature difference between the cooling object and its surroundings, which is what causes the discrepancy between Newton's Law of Cooling and what Mpemba discovered.

I shall attempt to find a relationship between the starting temperature of water and the time taken for it to cool. This relationship shall account for the Mpemba effect. To find this relationship, I must first find the relationship between the initial temperature and k.

Plotting a scatterplot of the starting temperature and k, I get the following graph.



Graph 2.1 – Relationship between k and starting temperature

I vertically stretched this graph by a factor of 1000000

The equation of best fit for this graph is $k = -0.0583{T_0}^3 + 7.66{T_0}^2 - 315T_0 + 4780$

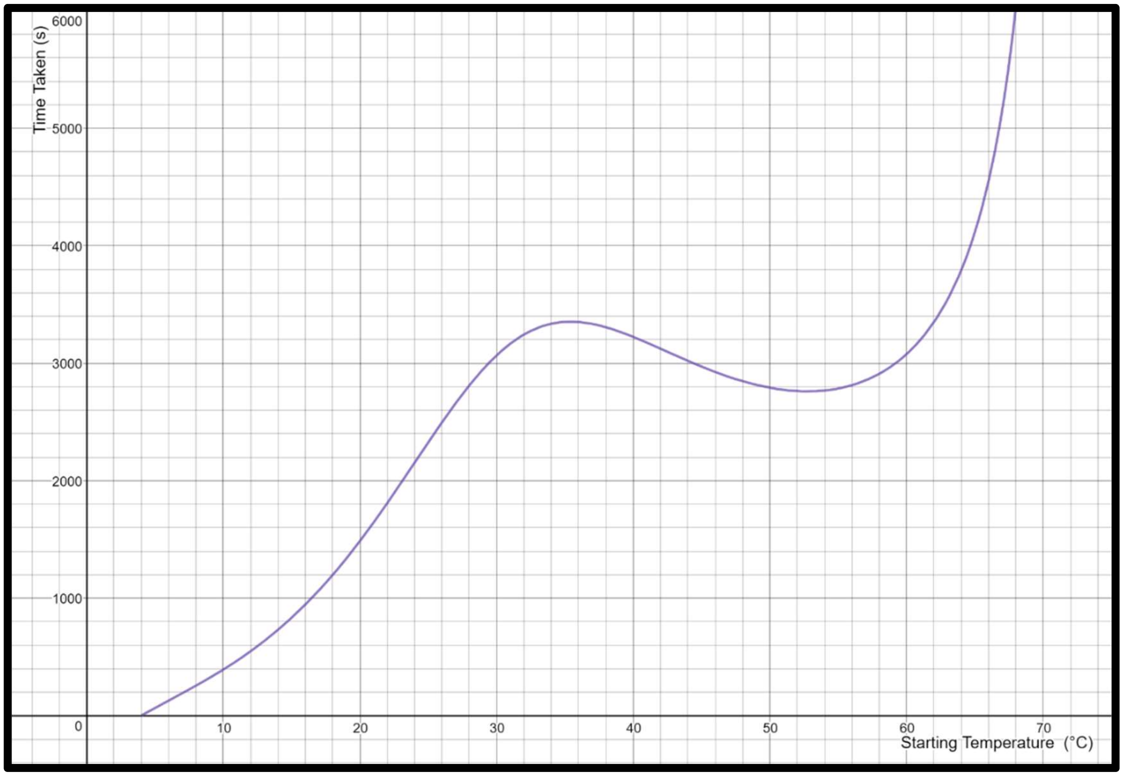
When plugging this into the original equation, I am left with

$$T = T_0 e^{-(0.000001)(-0.0583{T_0}^3+7.66\ {}_0^{\,2}-315T_0+47\quad)t}$$

To find the time taken for water to cool to 4°C for each starting temperature, I must plot the relation

$$4 = T_0 e^{-(0.000001)(-0.0583{T_0}^3+7.66{T_0}^2-31\quad {}_0+47\quad)t}$$

Where $T = 4, T_0$ is plotted on the x-axis and t is plotted on the y-axis



Graph 2.2 – Time taken to cool to 4°C depending on starting temperature

This graph above is similar to what is expected from Newton's Law of Cooling, but only up to 30°C, after which the kink in the graph causes a discrepancy. The kink in the graph shows that water starting at a higher temperature might cool faster than water starting at a lower temperature in specific cases. I assume that the Mpemba effect occurs for only a certain range of temperatures. In this experiment, I have found a relation between the starting temperature of water and the time taken for it to cool. This relation accounts for the Mpemba effect.

Evaluation

Although I successfully found an approximate model of relationship between the starting temperature of water and its rate of cooling, I must identify areas of improvement in the methodology of the experiment so that if one chooses to repeat it, they can arrive at a more accurate result.

While I attempted to control uncertainties, there are still a few things I could have done to make the experiment more precise. The amount of water in the beaker during certain trials before and after cooling differed – some of it must have evaporated. During one trial 4 grams of water evaporated, which might have had an impact on the rate of cooling and reduced the time taken for that sample to cool. To fix this issue, I could cover the beaker so that the water that evaporates, condenses later and falls back into the beaker, thus keeping the mass of water in the beaker constant. I think that the water that evaporated during my trials was minimal (4 grams out of 200 grams) and wouldn't have had a significant effect on the result of the experiment.

The fluctuations in the temperature of the freezer also caused a significant uncertainty in the experiment. If it were possible to prevent the freezer's internal temperature from fluctuating, that would allow for a more accurate experiment. However, a specialized freezer - in which the temperature doesn't naturally fluctuate – would have to be used. Being a high school student, I did not have access to this equipment and couldn't limit this uncertainty. The temperature fluctuations had a significant impact on the data collected. The time taken to cool to a certain temperature had uncertainties of up to ±270 seconds because of this source of random error.

Finally, the model found in Graph 2.1 breaks down at a starting temperature of 74°C. For temperatures below this, my model fits the data. I hypothesize that the line of best fit in Graph 2.1 would curve asymptotically towards the x-axis as x increases. This means that as the starting temperature increases, the value of k approaches 0. To confirm this, I would have to continue running trials with different initial temperatures - an extension to this experiment that I could perform in the future.

Conclusion

As demonstrated by the experiment, the Mpemba effect proves Newton's law of cooling wrong. The relationship between the starting temperature of water and its rate of cooling that I have found is

$$T = T_0 e^{-(0.000001)(-0.0583{T_0}^3+7.66{T_0}^2-315T_0+4780)t}$$

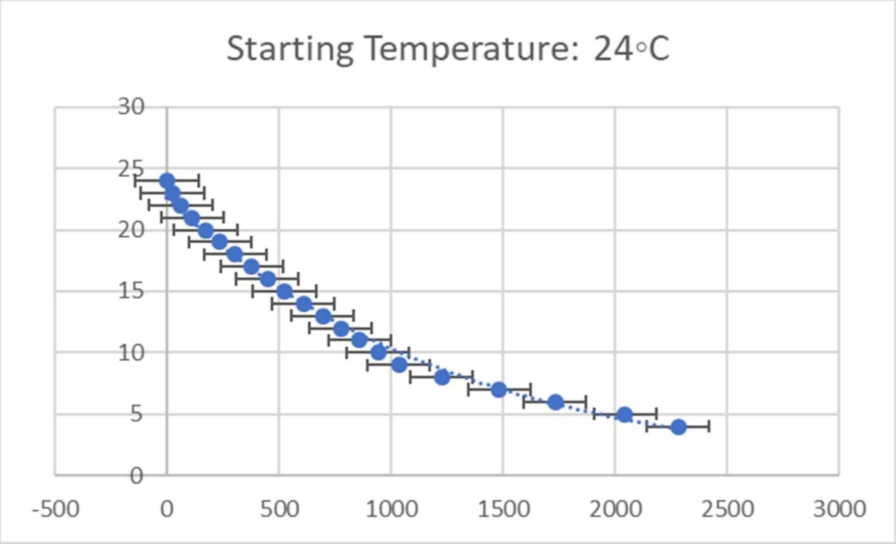
Some experiments have not observed the Mpemba effect, and I believe that this is because the range of initial temperatures they use for experimentation doesn't correspond to the range of initial temperatures at which the Mpemba effect is observed. Further experimentation would have to take this range into consideration.

There is still a lot of scope for further experimentation. I can choose to vary a whole list of other variables to arrive at a more detailed relationship between the initial temperature of water and its rate of cooling. The constants that I have found are only valid for the conditions of my experiment. For example, changing the total surface area of the container or the amount of water to be cooled would play a role in its rate of cooling. By varying these values, I could arrive at a more detailed equation that would take all these factors into account.

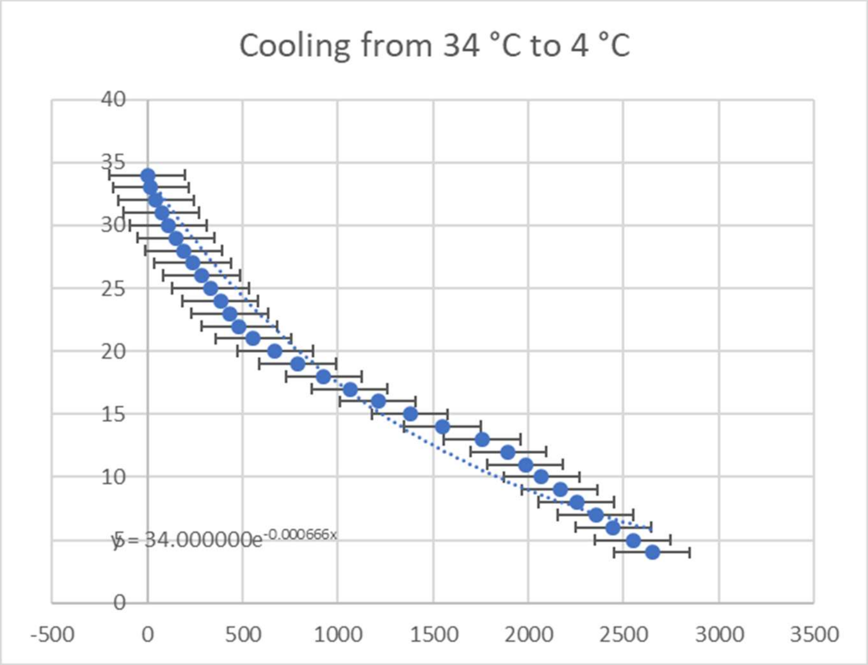
Appendix

All the raw data gathered from the experiment and the graphs (with uncertainties) of the averaged values can be found below.

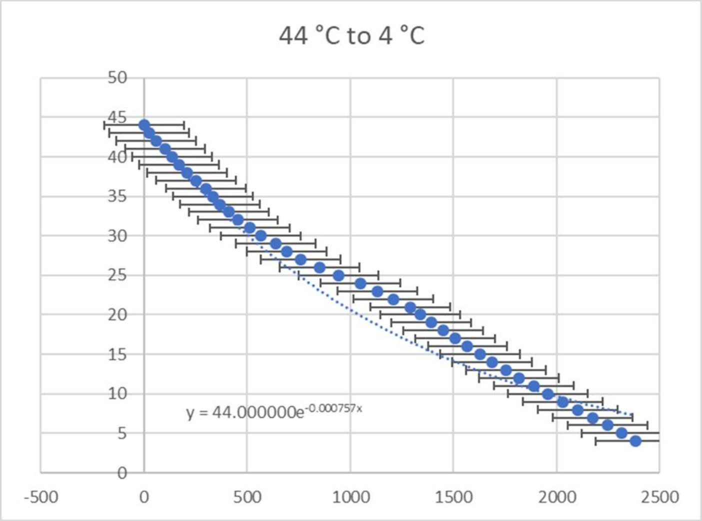
Starting Temperature - 24◦C				
Temperature (◦C)	Trial 1 (s)	Trial 2 (s)	Trial 3 (s)	Average Time (s)
24	0	0	0	0
23	20	28	24	24
22	55	70	61	62
21	96	128	115	113
20	147	208	164	173
19	189	296	223	236
18	234	387	297	306
17	281	487	369	379
16	351	566	430	449
15	427	645	502	525
14	512	725	593	610
13	599	808	684	697
12	680	876	772	776
11	763	965	855	861
10	846	1057	929	944
9	940	1142	1026	1036
8	1129	1345	1207	1227
7	1388	1629	1441	1486
6	1638	1875	1689	1734
5	1950	2191	2003	2048
4	2189	2426	2240	2285



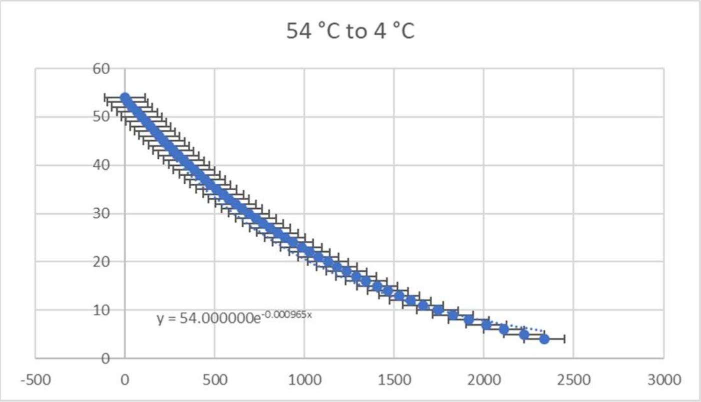
Starting Temperature - 34◦C				
Temperature (◦C)	Trial 1 (s)	Trial 2 (s)	Trial 3 (s)	Average Time (s)
34	0	0	0	0
33	19	17	18	18
32	47	37	48	44
31	80	71	74	74
30	113	96	109	109
29	158	136	152	149
28	204	182	186	191
27	248	224	237	236
26	295	275	282	284
25	348	329	319	332
24	392	371	383	382
23	430	434	432	432
22	477	485	483	482
21	533	551	581	555
20	598	673	739	670
19	641	789	937	789
18	693	926	1159	926
17	802	1065	1320	1062
16	960	1311	1359	1210
15	1128	1573	1433	1378
14	1355	1755	1540	1550
13	1508	1823	1946	1759
12	1646	1894	2142	1894
11	1731	1994	2224	1983
10	1818	2071	2318	2069
9	1913	2172	2410	2165
8	2008	2248	2503	2253
7	2102	2362	2596	2353
6	2197	2453	2689	2446
5	2293	2549	2805	2549
4	2393	2623	2932	2649



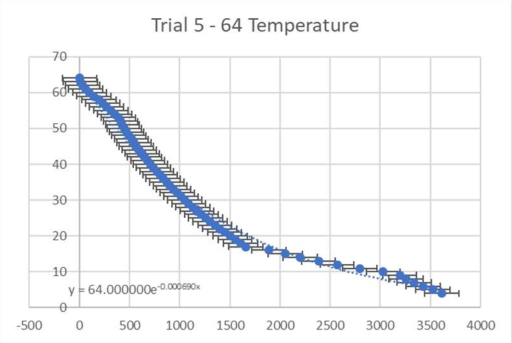
Starting Temperature - 44◦C				
Temperature (◦C)	Trial 1 (s)	Trial 2 (s)	Trial 3 (s)	Average Time (s)
44	0	0	0	0
43	21	23	28	24
42	58	61	64	61
41	95	98	110	101
40	129	134	151	138
39	162	167	178	169
38	193	204	224	207
37	238	243	275	252
36	271	291	338	300
35	300	324	375	333
34	334	358	409	367
33	389	403	444	412
32	426	448	497	457
31	464	506	575	515
30	501	555	636	564
29	555	630	732	639
28	609	683	784	692
27	676	749	849	758
26	768	842	943	851
25	861	935	1036	944
24	968	1041	1142	1050
23	1047	1121	1222	1130
22	1127	1201	1302	1210
21	1200	1280	1387	1289
20	1252	1331	1432	1340
19	1308	1384	1486	1393
18	1365	1442	1543	1451
17	1424	1499	1600	1508
16	1477	1559	1667	1568
15	1543	1619	1721	1628
14	1596	1677	1786	1686
13	1666	1743	1847	1752
12	1734	1810	1913	1819
11	1799	1879	1986	1888
10	1867	1946	2053	1955
9	1938	2020	2130	2029
8	2008	2091	2201	2100
7	2087	2164	2268	2173
6	2156	2236	2343	2245
5	2229	2304	2405	2313
4	2294	2373	2478	2382



Starting Temperature - 54°C				
Temperature (°C)	Trial 1 (s)	Trial 2 (s)	Trial 3 (s)	Average Time (s)
54	0	0	0	0
53	15	17	16	16
52	40	43	43	42
51	64	74	69	69
50	89	106	87	94
49	116	125	116	119
48	138	159	135	144
47	163	188	150	167
46	186	230	163	193
45	211	260	186	219
44	237	286	215	246
43	264	308	241	271
42	289	338	270	299
41	322	368	294	328
40	346	397	328	357
39	373	425	360	386
38	405	455	391	417
37	432	480	426	446
36	461	510	460	477
35	499	549	488	512
34	524	584	539	549
33	547	618	572	579
32	572	657	622	617
31	594	688	677	653
30	629	722	719	690
29	659	770	764	731
28	690	811	809	770
27	732	846	846	808
26	776	885	892	851
25	812	929	935	892
24	857	976	981	938
23	910	1018	1021	983
22	957	1070	1066	1031
21	1005	1115	1117	1079
20	1055	1171	1167	1131
19	1101	1222	1220	1181
18	1156	1271	1275	1234
17	1215	1326	1326	1289
16	1271	1386	1378	1345
15	1331	1439	1448	1406
14	1385	1502	1505	1464
13	1455	1563	1563	1527
12	1518	1626	1626	1590
11	1595	1695	1696	1662
10	1681	1777	1768	1742
9	1779	1861	1841	1827
8	1872	1948	1931	1917
7	1964	2047	2025	2012
6	2064	2142	2130	2112
5	2174	2256	2236	2222
4	2293	2369	2343	2335



Starting Temperature - 64◦C				
Temperature (◦C)	Trial 1 (s)	Trial 2 (s)	Trial 3 (s)	Average Time (s)
64	0	0	0	0
63	10	13	10	11
62	29	38	26	31
61	57	70	62	63
60	95	114	97	102
59	136	171	125	144
58	179	223	177	193
57	212	272	212	232
56	247	318	251	272
55	260	352	327	313
54	284	388	381	351
53	310	431	432	391
52	331	459	449	413
51	353	472	462	429
50	373	488	486	449
49	392	513	511	472
48	415	537	533	495
47	436	563	558	519
46	464	582	586	544
45	483	610	611	568
44	513	638	640	597
43	547	661	664	624
42	569	692	692	651
41	601	727	715	681
40	627	755	748	710
39	658	779	783	740
38	695	810	808	771
37	723	840	843	802
36	757	881	867	835
35	789	911	907	869
34	820	949	943	904
33	861	979	974	938
32	899	1018	1008	975
31	930	1054	1055	1013
30	968	1088	1094	1050
29	1013	1136	1121	1090
28	1049	1170	1177	1132
27	1091	1212	1213	1172
26	1134	1262	1252	1216
25	1177	1300	1303	1260
24	1229	1351	1335	1305
23	1270	1392	1391	1351
22	1323	1445	1429	1399
21	1371	1491	1482	1448
20	1422	1542	1539	1501
19	1470	1591	1595	1552
18	1528	1645	1651	1608
17	1578	1705	1700	1661
16	1805	1933	1926	1888
15	1973	2089	2094	2052
14	2122	2245	2242	2203
13	2303	2427	2422	2384
12	2487	2608	2615	2570
11	2713	2835	2834	2794
10	2947	3072	3071	3030
9	3115	3236	3231	3194
8	3178	3303	3296	3259
7	3256	3381	3386	3341
6	3351	3470	3469	3430
5	3445	3570	3566	3527
4	3534	3659	3655	3616



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[i] (Q & A: Freezing Hot Water and the Mpemba Effect.)
[ii] (Q & A: Freezing Hot Water and the Mpemba Effect.)
[iii] (Jeng)
[iv] (Other differential equations)
[v] (Newton's Law of Cooling.)
Graphs were drawn in DESMOS and Microsoft Excel.