IB Mathematics HL Internal Assessment

What is the risk of a fatal head injury during a car crash?

Aim and Rationale

The investigation aims to create a mathematical model for a car crash. Using this model, I will investigate the efficiency of airbags using the Head Injury Criterion. Recently, one of my friends was in a major car accident and somehow managed to walk away from it without a scratch. This event got me thinking about how car manufacturers design their cars with the safety of the driver and passengers in mind, while still maintaining the aesthetic appeal of the vehicle.

The city I live in has recorded a 45 percent increase in traffic fatalities in the first half of 2016, compared with the same period last year (Barakat). Car crashes are increasing at an alarming rate and car companies and government bodies such as the National Highway Traffic Safety Administration (NHTSA) run expensive tests to check how well each new car protects its occupants during a car crash. With a mathematical model, these companies could predict how a car would react in a crash and reduce the number of physical tests they would have to run, thus saving a lot of money. This model would also allow them to ensure that the car looks safe on paper before manufacturing prototypes and running various tests on them.

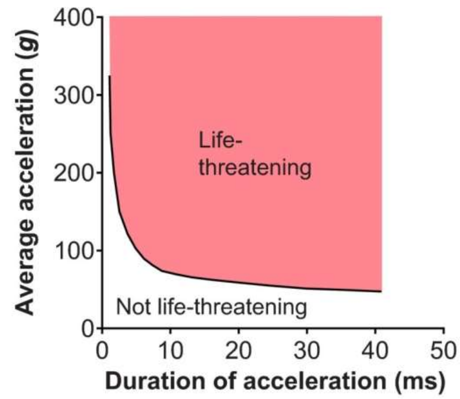
The Head Injury Criterion

When a car moves at a certain velocity, all its occupants are moving at the same velocity. And during a crash, its occupants undergo a large magnitude of deceleration (or negative acceleration) in a short amount of time. The car stops because a direct force was applied to it, but the occupants tend to continue moving forward (due to inertia of motion) till an opposing force stops them. This force could be provided by a seatbelt, airbag, or steering wheel. We know:

$$a = \frac{dv}{dt} = \frac{d^2s}{dt^2}$$

where a is acceleration, v is velocity, s is displacement, and t is time.

Data collected from crash tests revealed that the probability of receiving a head injury is strongly correlated to the duration of the crash and the magnitude of deceleration. A low average acceleration for a long time, or a high average acceleration for a very short time, isn't life-threatening. Once this correlation was found, the Wayne State curve was drawn with the equation $(\bar{a})^{2.5} \times T = 1000$. Using this equation and the graph on the right, we can conclude that if $(\bar{a})^{2.5} \times T > 1000$, then the car crash is life-threatening. Consequently, to prevent fatal injuries, the time during which the passenger decelerates must be increased. Airbags were



Wayne State Curve

installed in cars because they slow down the impact of a passenger's head by increasing the amount of time during which it decelerates.

To predict the risk of a head injury, car crash analysts created the Head Injury Criterion (HIC) based on the Wayne State Curve mentioned earlier. This criterion is a mathematical function that considers the duration of the crash and the magnitude of deceleration. With these two variables, it is possible to judge the risk of head injury quantitatively. HIC is defined as

$$HIC = \left\{ \left[\frac{1}{t_2 - t_1} \int_{t_1}^{t_2} a(t)dt \right]^{2.5} (t_2 - t_1) \right\}_{max}$$

where a is acceleration and t_1 and t_2 are the initial and final times (in seconds) of the interval during which HIC attains a maximum value. "This interval is limited to a specific value between 3 and 36 milliseconds" (McHenry).

"At a HIC of 1000, there is an 18% probability of a severe head injury, a 55% probability of a serious injury and a 90% probability of a moderate head injury to the average adult" (Mackay).

Data Collection

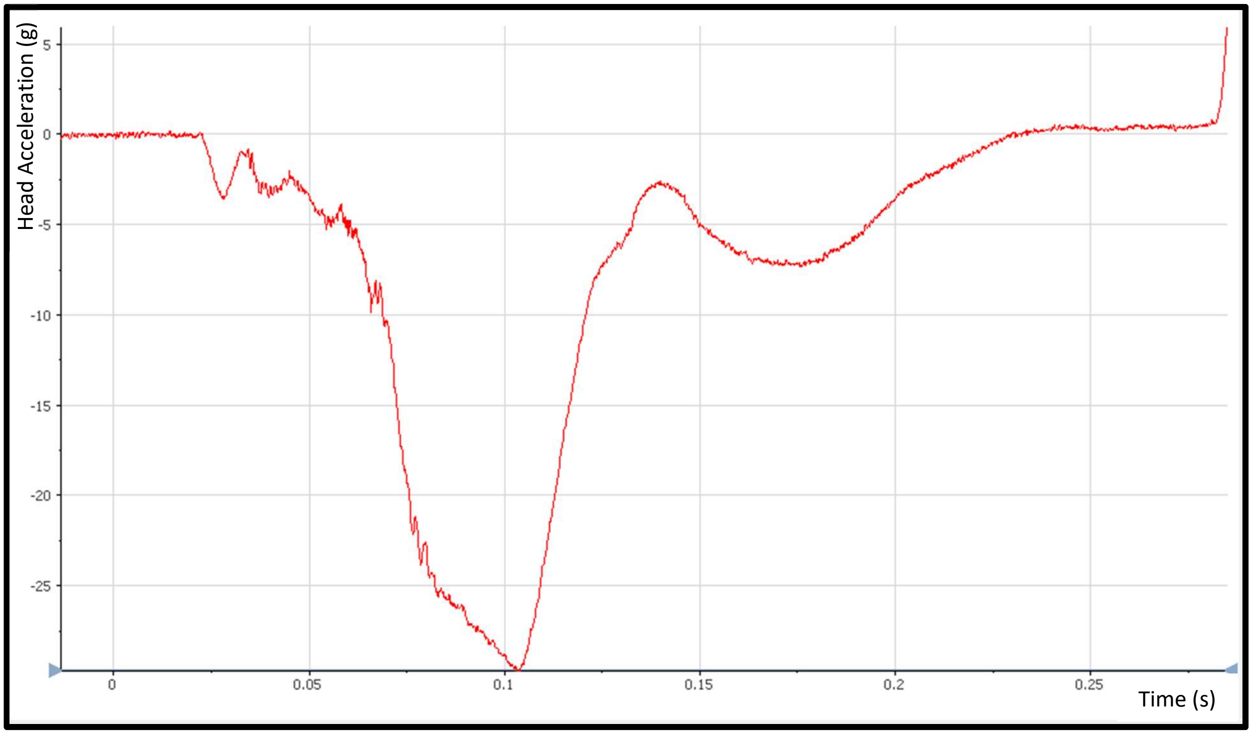
I will model a car crash with real-world data. Since I couldn't perform the experiment myself, I contacted the Insurance Institute for Highway Safety (IIHS) and am using their experimental data (found in a separate document). "The IIHS conducts vehicle tests to determine crashworthiness — how well a vehicle protects its occupants in a crash" (Top Safety Picks by Year). I chose to analyze the data collected from the crash tests of the Honda CR-V because it was the car that my friend was driving. I was also intrigued by this car because it was awarded the Top Safety Pick by the IIHS and I wanted to mathematically investigate how safe this car was.

I am investigating a frontal crash where the airbags were deployed as this is the type of accident that my friend was involved in and because it is one of the most common types of car accidents that results in fatalities. The dataset that I will be working with is the one of acceleration of the occupant's head. With this dataset, I will be able to plot the graph of acceleration against time, model a function for the curve, and calculate the head injury criterion. The Moderate Overlap Front Test was carried out on the 2017 Honda CR-V (CEF1702) weighing 1,693 kg traveling at 64.4 km/h (Insurance Institute for Highway Safety).

Using the software DIAdem, a data visualization, management, and analysis tool by National Instruments, I can plot the data collected by their sensors and perform various mathematical functions on it, such as curve fitting and differentiation of the curve.

Modeling the Curve

The graph of the acceleration of the head during the crash is shown below. The crash occurs between 0 and 0.25 seconds. The acceleration of the graph is in g-forces. G-forces are units of acceleration which are multiples of the force of gravity (1 g is 9.81 m/s^2). To derive a function from this graph to perform operations on, I shall focus on the time interval between 0.08 and 0.12 seconds. I chose this section of the curve as it closely resembles a parabola and is the time during which the occurrence of a head injury is most probable.



Graph of acceleration of head along x-axis

I need to find a function that represents this crash and to do so, I need to select a curve-fitting method. Initially, I tried doing so with a piecewise function. However, when I integrated this function, the graph was no longer continuous, and I couldn't perform the necessary calculations. I then did more research on mathematical methods to approximate functions and finally chose to find the Taylor Polynomial of this data because I only needed an approximation over a certain interval. Also, the software DIAdem could give me higher order derivates, thus allowing me to easily find the polynomial. I couldn't find the derivatives manually because I only had a relation, not a function, in terms of x and y representing the crash data.

Finding the Taylor Polynomial is a method of approximating a function within a given interval about $x = a$ by only using derivative information at that point. This polynomial would allow me to perform various functions such as finding the Head Injury Criterion. The method of finding a Taylor Polynomial isn't part of the Mathematics HL Core Syllabus nor part of Option Statistics, so I had to investigate it myself. I found that it was part of the Calculus Option and learnt it from there. Soon, I shall explain why the Taylor Polynomial acts as a good approximation of the data. The general equation of the Taylor Polynomial is written below.

$$T_n(x) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i$$

Note that the little i notation on the f means the derivative of order i (it doesn't mean f to the i power, even though the little i seems to look like an exponent) (Ault).

It is apparent from this equation that I need to find higher order derivatives of the curve at $x = a$. In DIAdem, I first applied a Savitzky-Golay filter on the data to smooth the curve and prevent random measurement errors from affecting the calculations. This filter "is a method of data smoothing based on local least-squares polynomial approximation" (Schafer). The value of a I chose is 0.0985 because this is the x-coordinate of the point of inflection of the curve.

At $\boldsymbol{a = 0.0985\ s}$		
Function	$f(a)$	-28.06
First Derivative	$f'(a)$	0
Second Derivative	$f''(a)$	60527
Third Derivative	$f'''(a)$	5365394
Fourth Derivative	$f''''(a)$	-651661754

Because the accuracy of these measurements is varied, the number of significant figures for the values will differ. After all the calculations, I shall round my answer to three significant figures.

I haven't included values after the fourth derivative because the function approximated the graph well and adding more coefficients showed negligible difference in between the interval of 0.08s to 0.12s.

I want to explore the reason why the Taylor Polynomial works, so the subsequent steps investigate the equation. All of the steps below are centered around $x = a$. Taking a general equation $v(x)$

$$v(x) = b + c(x-a) + d(x-a)^2 + e(x-a)^3 + f(x-a)^4$$

To approximate the curve at a, I first need the value of y of the two graphs to be equal at this point. At $x = a$, all terms of the polynomial, other than the constant, equate to 0. Hence, $v(x) = b = -28.06$. I also want the slope of the function to be equal to the slope of the curve at this point. To find the slope, I take $v'(x)$.

$$v'(x) = c + 2d(x-a) + 3e(x-a)^2 + 4f(x-a)^3$$

Again at $x = a$, all terms other than the constant equate to 0. Hence, $v'(x) = c = 0$. a is the point of inflection of the graph so it is clear that the first derivative at this point would be 0. Additionally, I want the rate of change of the slope of the function to be the same at this point. To find this, I take $v''(x)$.

$$v''(x) = 2d + 6e(x-a) + 12f(x-a)^2$$

Again at $x = a$, all terms other than the constant equate to 0. Hence, $v''(x) = 2d = 60527$ and $d = \frac{v''(x)}{2}$. Repeating this process for higher order derivatives gives me $e = \frac{v'''(x)}{6}$ and $f = \frac{v''''(x)}{24}$. I noticed that the denominator is simply the factorial of the order of derivative that I am taking. And this gives us the general equation of the Taylor Polynomial, which, when expanded to the fourth order derivative, is

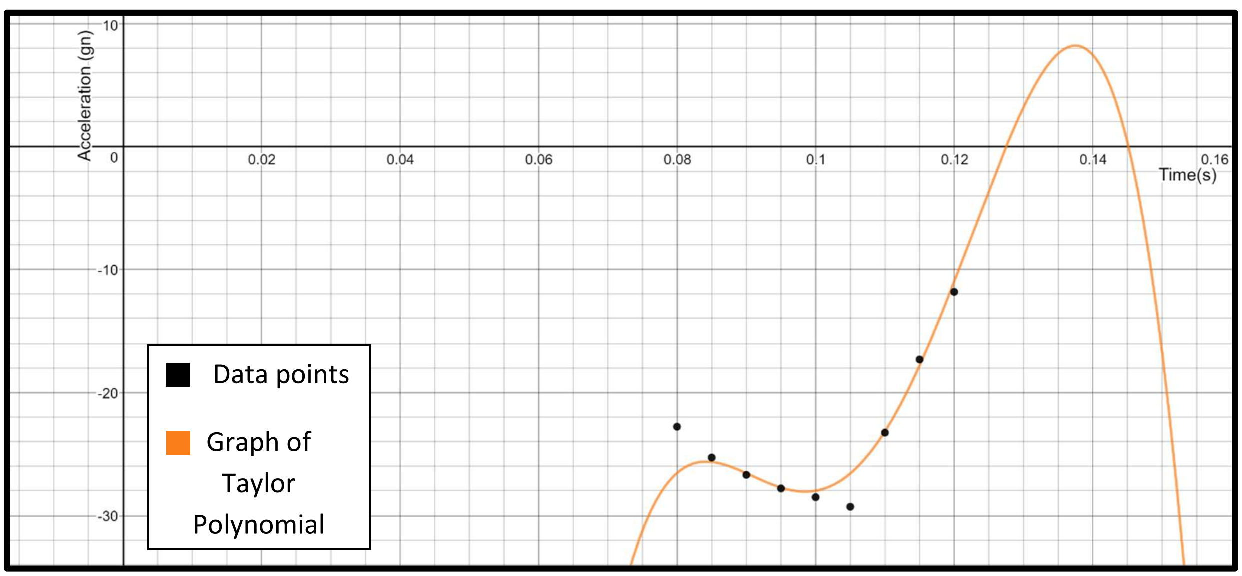
$$T_n(x) = f(a) + \frac{f'(a)}{1}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \frac{f''''(a)}{4!}(x-a)^4$$

Substituting the values of higher order derivatives, I get the following equation.

$$T_n(x) = -28.06 + \frac{60527}{2!}(x-0.0985)^2 + \frac{5365394}{3!}(x-0.0985)^3 + \frac{-65166175}{4!}(x-0.0985)^4$$

And simplifying the terms gives me a final equation which I have graphed below.

$$T_n(x) = -28.06 + 30263.5(x-0.0985)^2 + 894232(x-0.0985)^3 - 27152573(x-0.0985)^4$$



Acceleration data points and Taylor Polynomial

This is the function of the relationship between time (x) and acceleration of the occupant's head $T_n(x)$. As we can see in the graph above, it fits the data quite well in the interval 0.08s to 0.12s. I could simplify the equation further by expanding the terms, but this form allows for easy substitution.

Finding the HIC

I can now use this function to find the Head Injury Criterion of the Honda CRV. As a reminder, the formula for HIC is

$$HIC = \left\{\left[\frac{1}{t_2 - t_1}\int_{t_1}^{t_2} a(t)dt\right]^{2.5} (t_2 - t_1)\right\}_{max}$$

Note that $a(t) = T_n(x)$. Substituting the Taylor Polynomial and changing the variables, I get

$$HIC = \left\{\left[\frac{1}{t_2 - t_1}\int_{t_1}^{t_2} (-28.06 + 30263.5(t - 0.0985)^2 + 894232(t - 0.0985)^3 - 27152573(t - 0.0985)^4)dt\right]^{2.5} (t_2 - t_1)\right\}_{max}$$

The first step is to find $\int_{t_1}^{t_2} a(t)dt$ over different intervals. The intervals (correct to 3 decimal places) that I shall be using for this calculation are

t_1	t_2
0.075	0.125
0.080	0.120
0.085	0.115
0.090	0.110

For the interval $t_1 = 0.080$ and $t_2 = 0.120$

$$\int_{0.080}^{0.120} (-28.06 + 30263.5(t - 0.0985)^2 + 894232(t - 0.0985)^3 - 27152573(t - 0.0985)^4)dt$$

One amazing characteristic of the Taylor Polynomial is that it allows for simple integration. All I have to do here is apply the reverse power rule.

$$= [-28.06t + 10087.3(t - 0.0985)^3 + 223558(t - 0.0985)^4 - 5430514.6(t - 0.0985)^5]_{0.080}^{0.0120}$$

Now I shall find the definite integral by substituting the upper and lower bound

$$= |((-28.06 \times 0.0120) + 10087.3(0.0120 - 0.0985)^3 + 223558(0.0120 - 0.0985)^4 - 5430514.6(0.0120 - 0.0985)^5) - ((-28.06 \times 0.080) + 10087.3(0.080 - 0.0985)^3 + 223558(0.080 - 0.0985)^4 - 5430514.6(0.080 - 0.0985)^5)|$$

$$= |-0.973| = 0.973$$

Using this value in the HIC formula, I can find the Head Injury Criterion during this time period.

$$(0.040)\left[\frac{1}{0.040} \times 0.973\right]^{2.5} = 116.85 \approx 117\ (3SF)$$

Similarly, using technology, I substituted the other values of t_1 and t_2 to find maximum HIC.

t_1	t_2	$\int_{t_1}^{t_2} a(t)dt$	$\left[\frac{1}{t_2 - t_1}\int_{t_1}^{t_2} a(t)dt\right]^{2.5} (t_2 - t_1)$
0.075	0.125	1.152	127
0.080	0.120	0.973	117
0.085	0.115	0.771	101
0.090	0.110	0.538	75.1

The maximum value of HIC I have found in this interval is 127, but there are infinitely many intervals that can be considered. Hence, I conclude that the value of HIC is approximately 130.

$$HIC = \left\{\left[\frac{1}{t_2 - t_1}\int_{t_1}^{t_2} a(t)dt\right]^{2.5} (t_2 - t_1)\right\}_{max} \approx 130$$

Conclusion

After finding a Taylor Polynomial to represent the car crash and calculating the HIC of the Honda CRV as approximately 130 when it is traveling at 64.4 km/h, it is clear that the Honda CRV is a very safe car and there is no doubt that it deserved the Top Safety Pick by the IIHS. The value I found is a close match to the one calculated by IIHS, which was 135.47. By modeling different types of car crashes, such as from the side and rear of the car, I could truly determine how safe the car is in any situation.

With HIC information, I can compare cars from competing companies to see how they fare against each other. When I performed the same calculations on crash data of the Tesla Model S (CEF1611), I found a HIC value of approximately 600 (the IIHS calculated the value as 604). Even though this is well below 1000 - the point at which it is very likely that a fatal injury will occur –

the IIHS gave Tesla an “acceptable” rating. Consequently, Tesla improved their safety precautions before releasing the car on the public market.

While the Honda CRV is approximately 5 times safer than the Tesla Model S (looking at HIC values only), the Model S has features like artificial intelligence which attempt to prevent crashes altogether. The effectiveness of these additional features must be taken into account when comparing the safety between two cars, but we must keep in mind that they do not help in the event of a crash.

In this investigation, I have created a mathematical model for a car crash, understood the significance of the Head Injury Criterion, calculated its value for the Honda CR-V, and used it to compare the safety of this car to a Tesla Model S.

Bibliography

Ault, Shaun. *Taylor Polynomial: Formula & Examples*. n.d. 18 February 2018. <https://study.com/academy/lesson/taylor-polynomial-formula-examples.html>.

Barakat, Noorhan. *112 traffic deaths in six months in Dubai*. 7 August 2016. 18 February 2018. <http://gulfnews.com/news/uae/transport/112-traffic-deaths-in-six-months-in-dubai-1.1875132>.

Henn, Hans-Wolfgang. "Crash Tests and the Head Injury Criterion." *Teaching Mathematics and its Applications* 17.4 (1998).

Insurance Institute for Highway Safety. *IIHS TechData*. n.d. 18 February 2018. <https://techdata.iihs.org/>.

Lindgren, Sten, ed. *Modern Concepts in Neurotraumatology*. Springer, Vienna, n.d. <doi.org/10.1007/978-3-7091-8859-0>.

Mackay, Murray. "The Increasing Importance of the Biomechanics of Impact Trauma." *Sadhana Academy Proceedings in Engineering Sciences* 32.4 (2007): 397-408. <link.springer.com/article/10.1007/s12046-007-0031-9>.

McHenry, Brian G. "HIC and the ATB." 2004. 18 February 2018. <www.mchenrysoftware.com/HIC%20and%20the%20ATB.pdf>.

Mellander, H. *HIC—the Head Injury Criterion*. Springer Vienna, 1986.

Schafer, Ronald W. "What Is a Savitzky-Golay Filter?" *IEEE SIGNAL PROCESSING MAGAZINE* (2011). 18 February 2018. <https://inst.eecs.berkeley.edu/~ee123/sp18/docs/SGFilter.pdf>.

Top Safety Picks by Year. 2016. 18 February 2018. <www.iihs.org/iihs/ratings/TSP-List/2016>.

"Wayne State Tolerance Curve." n.d. *National Institutes of Health.* Image. <https://openi.nlm.nih.gov/imgs/512/331/3820257/PMC3820257_DMM011320F2.png?keywords=head+injuries>.
